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Numerical Computation of Korteweg-de Vries (KdV) Equation Using Finite Difference Approximation

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Abstract
In this paper, we study a general form of third order nonlinear partial differential equation known as Korteweg-de Vries (KdV) equation. A traveling wave solution method is discussed for analytic solution of the general form of KdV equation. In order to understand the effect of convection and dispersion terms of the equation we present a numerical evaluation of the analytical solution for various values of convection and dispersion coefficients. Finite difference scheme for the numerical solution of the KdV equation is investigated and stability condition for a first-order scheme using convex combination method is determined. Von Neumann stability analysis is performed to determine the stability condition for a second order scheme. We present error estimation of the numerical schemes and verify qualitative behavior of the KdV equation.

Keywords: KdV Equation, Non-Conservative Form, Solitary Wave, Finite Difference Scheme.

1. Introduction

Waves are created on the sea or profound surface and naturally, they are playing an attractive and noticeable phenomenon that impacts every aspect of life on the earth. As for this, one needs to include this winsome experience into mathematical models and analysis. In the year of 1895, before all, Korteweg and de Vries [1] developed the Korteweg-de Vries (KdV) equation to model weakly nonlinear waves. This KdV equation has been used in various fields such as water wave [2], plasma physics [3], bubble-liquid mixture [4] and so on. It is also apposite to pulse wave propagation in blood vessels. The solution of the KdV equation is identical to as soliton, and it is recently found that signals carry within neurons in the form of solitons [5,6,7]. These solitons may take place in proteins and DNA (deoxyribonucleic acid) [8] where they are related to low-frequency collective motion.

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Zabusky and Kruskal [9] obtained the numerical solution of the KdV equation in the year of 1965. Furthermore to solve the KdV equation numerically, there are several methods have been discussed in various paper. Investigation of Finite Difference Method (FDM) with sufficient accuracy for the KdV equation was undertaken. First-order and second-order scheme in non-conservative forms for the KdV equation is studied for investigation. A second-order ZK finite difference scheme (ZK scheme) is preferred by Zabusky and Kruskal in [10]. This scheme is considered for non-conservative form and the convection velocity is calculated by the average of the three neighboring grid points. However, in [14] we found a second order conservative form shows more accurate result than ZK non-conservative scheme [10]. In this paper, we perform a comparative study to understand the accuracy of the second order non-conservative form of the KdV equation. Here also we will discuss the traveling wave solution method for the analytical solution of the general form of the KdV equation.

At first, the analytic solution of the KdV equation is discussed. Then explicit finite difference schemes for the numerical solution of the KdV equation is investigated and the stability conditions for the first and second order schemes are determined. Zabusky and Kruskal scheme is also presented in the same section. After that, the verification of the effects of convection and dispersion terms is discussed. Numerical results and explanation of graphical representations for various cases are discussed sequentially. At the end, some references are given.

2. Analytic Solution of the KdV Equation

The third order general form of nonlinear KdV equation is

$$\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} + \nu \frac{\partial^3 u}{\partial x^3} = 0. \qquad (1)$$

Here μ and ν are represent nonlinear and dispersion coefficients respectively. The equation has two separate terms: one is convection term and other is dispersion term. Here nonlinear term with nonlinear coefficient is called convection term and third order term with dispersion coefficient illustrating dispersion term. Analytic or exact solution of the KdV equation is discussed based on [11,12] where use a traveling wave solution method for finding the solution.

Consider a traveling wave solution of the form $u(x,t) = f(\xi)$ for $\xi = x - c\tau$, and $\tau = t$ where, c is the speed of the wave. Therefore, $u(x,t) = f(x - ct)$.

Now, we can write $\frac{\partial u}{\partial t} = -cf'(\xi)\,, \frac{\partial u}{\partial x} = f'(\xi)\,, \frac{\partial^3 u}{\partial x^3} = f'''(\xi)$. Substituting all these in Equation (1), and then integrating with respect to ξ, we get $-cf + \frac{\mu}{2}f^2 + \nu f'' = C_1$; where C_1 is integration constant. Here this constant goes to be zero due to decay conditions of f, we have $-cf + \frac{\mu}{2}f^2 + \nu f'' = 0$.

Now multiplying by f' and integrating we get $-\frac{c}{2}f^2 + \frac{\mu}{6}f^3 + \frac{\nu}{2}(f')^2 = C_2$; where the integrating constant C_2 becomes zero as for same condition, and we have

$-\frac{c}{2}f^2 + \frac{\mu}{6}f^3 + \frac{\nu}{2}(f')^2 = 0 \Rightarrow \frac{\nu}{2}(f')^2 = \frac{c}{2}f^2 - \frac{\mu}{6}f^3$

$\Rightarrow (f')^2 = f^2(\frac{c}{\nu} - \frac{\mu}{3\nu}f)$.

The real solution exist only if $(f')^2 \geq 0$, that is, $(\frac{c}{\nu} - \frac{\mu}{3\nu}f) \geq 0$.

Using separation variable, we get

$$\frac{df}{d\xi} = \pm f\sqrt{\left(\frac{c}{\nu} - \frac{\mu}{3\nu}f\right)} \Rightarrow \frac{df}{f\sqrt{\left(\frac{c}{\nu} - \frac{\mu}{3\nu}f\right)}} = \pm d\xi \qquad (2)$$

To evaluate the integral on the left hand side of (2) , we substitute $f = \frac{3\nu}{\mu}.\frac{c}{\nu}sech^2(w)$ and we get $df = -\left(\frac{6\nu}{\mu}.\frac{c}{\nu}\right)sech^2(w)\tanh(w)dw$.

From (2): $\int \frac{df}{f\sqrt{(\frac{c}{\nu} - \frac{\mu}{3\nu}f)}} = -2\sqrt{\frac{c}{\nu}}w$ and $-2\sqrt{\frac{\nu}{c}}w = \pm(\xi - \xi_0)$.

Therefore,

$$f(\xi) = \frac{3c}{\mu}sech^2\left[\frac{1}{2}\sqrt{\frac{c}{\nu}}(x - ct - \xi_0)\right]$$

$$\Rightarrow u(x,t) = \frac{3c}{\mu}sech^2\left[\frac{1}{2}\sqrt{\frac{c}{\nu}}(x - ct - \xi_0)\right] \qquad (3)$$

Here speed propagation of the wave is perform by c, this is proportional to the amplitude $\frac{3c}{\mu}$ that is, speed propagation is linearly related to the amplitude. For getting real and positive solution also for right movement c

goes to be positive $(c > 0)$. ξ_0 represents propagating constant. For increasing ξ_0 wave propagates left to right and wave moves right to left for decreasing ξ_0. The action of the dispersion and nonlinear terms are observed from exact solution. The width of the wave gradually spread out for gradual increase of the dispersion coefficient and reverse effect founded for decreasing, which is the effect of dispersion term. On the other hand, if value of the nonlinear coefficient is step by step increased then amplitude of the wave is gradually decreasing and for decreasing nonlinear coefficient, amplitude of the wave increasing which is effect of nonlinear term.

3. Finite Difference Methods of the KdV Equation

In this section, numerical solution for the KdV equation is investigated by explicit finite difference schemes. Zabusky and Kruskal presented a second-order explicit finite difference scheme (ZK scheme) for the KdV equation in [10] where the scheme is considered in non-conservative form and the convection velocity is considered the average of the three neighboring grid points. For further investigation first and second order schemes for non-conservative form is studied. Use convex combination method to find the stability condition of the first order scheme and Von Neumann stability analysis is presented for second order scheme. Here equal grid size is taken into consideration in these schemes.

3.1. (i) First Order Scheme with Stability Condition

Forward discretization of the time derivative, a backward discretization of the first order space derivative and second-order central difference in third order space derivative (FTBSCS technique) is considered to obtain the first-order scheme. The discrete form of KdV Equation (1) reads as:

$$u_j^{n+1} = u_j^n - \frac{\mu\Delta t}{\Delta x} u_j^n\left(u_j^n - u_{j-1}^n\right) - \frac{\nu\Delta t}{2(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right). \qquad (4)$$

This is an explicit finite difference scheme for the KdV equation in non-conservative form.

For determining the stability condition of the first order scheme (non-conservative form) use convex combination method. Here consider

$$\lambda_1 = \frac{\mu\Delta t}{\Delta x} \max_{n,j}\{u_j^n\} = \frac{\mu\Delta t}{\Delta x} \max_{j}\left\{u_j^0\right\} \text{ and } \lambda_2 = \frac{\nu\Delta t}{2(\Delta x)^3}.$$

Now Scheme (4) reads as:

$u_j^{n+1} = u_j^n - \lambda_1(u_j^n - u_{j-1}^n) - \lambda_2(u_{j+2}^n - \quad 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n)$
$\Rightarrow u_j^{n+1} = (1-\lambda_1)u_j^n + (\lambda_1 - 2\lambda_2)u_{j-1}^n + \ 2\lambda_2 u_{j+1}^n - \lambda_2 u_{j+2}^n + \lambda_2 u_{j-2}^n$
From above all the coefficients are positive and sum them is one so convex combination is applicable. Neumann Boundary Conditions $u_{j-2}^n = u_{j-1}^n$ and $u_{j+2}^n = u_{j+1}^n$ are used.
Therefore $u_j^{n+1} = (1-\lambda_1)u_j^n + (\lambda_1 - \lambda_2)u_{j-1}^n + \lambda_2 u_{j+1}^n$

Since sum of all coefficients is one, then by the rule of convex combination, $0 \leq (1-\lambda_1) \leq 1$; $0 \leq (\lambda_1 - \lambda_2) \leq 1$ and $0 \leq \lambda_2 \leq 1$
And hence the stability conditions are $\lambda_1 \leq 1$ and $\lambda_1 \geq \lambda_2$, where $\lambda_1, \lambda_2 \geq 0$.
From above conditions we have two relations

$$\Delta t \leq \frac{\Delta x}{\mu * \max_j \{u_j^0\}} \text{ and } \nu \leq 2\mu(\Delta x)^2 * \max_j \{u_j^0\}$$

3.2. (ii) Second Order Scheme with Stability Condition

Second order central difference in both time and space derivatives (CTCS technique) are performed for second order scheme. Then the KdV Equation (1) reads as:

$$u_j^{n+1} = u_j^{n-1} - \frac{\mu\Delta t}{\Delta x} u_j^n (u_{j+1}^n - u_{j-1}^n) - \frac{\nu\Delta t}{(\Delta x)^3}(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n). \qquad (5)$$

This is second-order explicit central difference scheme for KdV equation in non-conservative form.
Von Neumann stability analysis [11] is used for finding the stability condition of the second order non-conservative scheme. Consider $u_j^n = \xi^n e^{ikj\Delta x}$, insert Scheme (5)

$$\xi^{n+1} e^{ikj\Delta x} = \quad \xi^{n-1} e^{ikj\Delta x} - \frac{\mu\Delta t}{\Delta x} u_{max}\left(\xi^n e^{ik(j+1)\Delta x} - \xi^n e^{ik(j-1)\Delta x}\right) - \frac{\nu\Delta t}{(\Delta x)^3}\left[\xi^n e^{ik(j+2)\Delta x} - 2\xi^n e^{ik(j+1)\Delta x} + 2\xi^n e^{ik(j-1)\Delta x} - \xi^n e^{ik(j-2)\Delta x}\right]$$

where $u_j^n = \max_{n,j}\{u_j^n\} = \max_j \{u_j^0\} = u_{max}$.
Canceling $u_j^n = \xi^n e^{ikj\Delta x}$ from both sides, using Euler's formula, and letting $k\Delta x = \omega$, then

$$\xi = \xi^{-1} - \frac{\mu\Delta t}{\Delta x} u_{max}(2i * \sin(\omega)) - \frac{\nu\Delta t}{(\Delta x)^3}[2i * \sin(2\omega) - 4i * \sin(\omega)]$$

Let $A = \frac{2\mu\Delta t}{\Delta x} u_{max}(\sin(\omega)) + \frac{2\nu\Delta t}{(\Delta x)^3}[\sin(2\omega) - 2\sin(\omega)]$ (6)

Therefore $\xi = \xi^{-1} - iA$ is obtained, upon which multiplication by ξ,

$$\xi^2 + iA\xi - 1 = 0.$$

Now using quadratic formula

$\xi = \frac{\sqrt{4-A^2}}{2} - \frac{A}{2} i$ for $4 - A^2 \geq 0$, that is, $|A| \leq 2$ is obtained.

Consequently $|\xi| = \sqrt{\frac{4-A^2}{4} + \frac{A^2}{4}}$, which implies that $|\xi| = 1$.

From (6) $A = \frac{2\mu\Delta t}{\Delta x} u_{max}(\sin(\omega)) + \frac{2\nu\Delta t}{(\Delta x)^3}[\sin(2\omega) - 2\sin(\omega)]$.

To obtain maximum value which A attains, let $y = \sin(2\omega) - 2\sin(\omega)$ and solve for the value of ω for which $\frac{dy}{d\omega} = 0$, that is,

$$\frac{dy}{d\omega} = 4cos^2(\omega) - 2\cos(\omega) - 2$$

Therefore, $2cos^2(\omega) - \cos(\omega) - 1 = 0$,

$\omega = 0$ or $\omega = \frac{2\pi}{3}$, since $\omega \in [0, \pi]$.

Now when $\omega = 0$, $y = 0$ and also $A = 0$.

For $\omega = \frac{2\pi}{3}$, we have $y = -\frac{3\sqrt{3}}{2}$ and $|A| = \left|\frac{\sqrt{3}\mu\Delta t}{\Delta x} u_{max} - \frac{3\sqrt{3}\nu\Delta t}{(\Delta x)^3}\right|$.

For stability $|A| \leq 2$ and so the stability region satisfy the inequality

$$\frac{\Delta t}{\Delta x} \leq \frac{2}{\sqrt{3}} * \frac{1}{\left|u_{max} * \mu - \frac{3\nu}{(\Delta x)^2}\right|} .$$

3.3. (iii) Zabusky and Kruskal Scheme and Stability Condition

Zabusky and Kruskal (ZK) scheme is derived by central difference approximations for both space and time and the Equation (1) is a follows

$$u_j^{n+1} = u_j^n - \frac{\mu\Delta t}{3(\Delta x)}\left(u_{j+1}^n + u_j^n + u_{j-1}^n\right)\left(u_{j+1}^n - u_{j-1}^n\right) - \frac{\nu\Delta t}{(\Delta x)^3}\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right) . \quad (7)$$

In fact, this scheme is a modification of the Scheme (5), where the convection velocity u_j^n is taken by $\frac{\left(u_{j+1}^n + u_j^n + u_{j-1}^n\right)}{3}$. This scheme is a three-level scheme and second-order accurate in time. The truncation error is of order $(o(\Delta t)^2 + o(\Delta x)^2)$. The linear stability condition for this scheme is following to (subsection ii). The stability condition is as follows:

$$\frac{\Delta t}{\Delta x} \leq \frac{2}{\sqrt{3}} * \frac{1}{\left| u_{max} * \mu - \frac{3\nu}{(\Delta x)^2} \right|}$$

Where u_{max} is the maximum value of u depending on the amplitude of solitons.

4. Verification of the Effect of Convection and Dispersion Terms

In exact solution the effect of the nonlinear and dispersion term already mentioned. Now these effect are need to verify by numerical solution. In this section, for the justification of the both terms we use Scheme (5). At a fixed time, $t = 0.6$ and for different values of μ and , we present the both effect where $\Delta t = 0.001$ and $\Delta x = 0.2$ are used. In this case, space -10 to 10 and time 0 to 1 are considered. For fixed ν $(= 1)$, we change the value of μ $(= 6, 5, 4)$ taking respectively, which represents the effect of convection term. Again, for fixed μ $(= 6)$, we take the values of ν $(= 1, 2, 3)$ respectively, which represents the effect of the dispersion term.

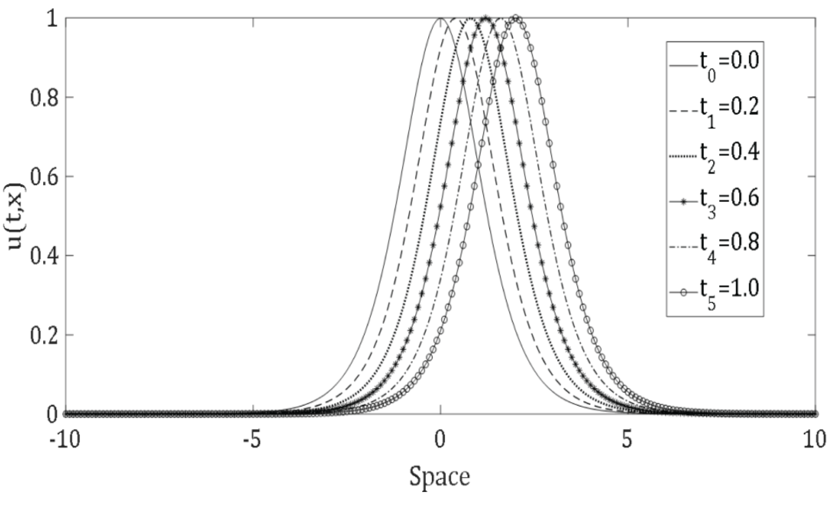

Figure 1. Analytical Solution of the KdV Equation for $\mu = 6$ and $\nu = 1$.

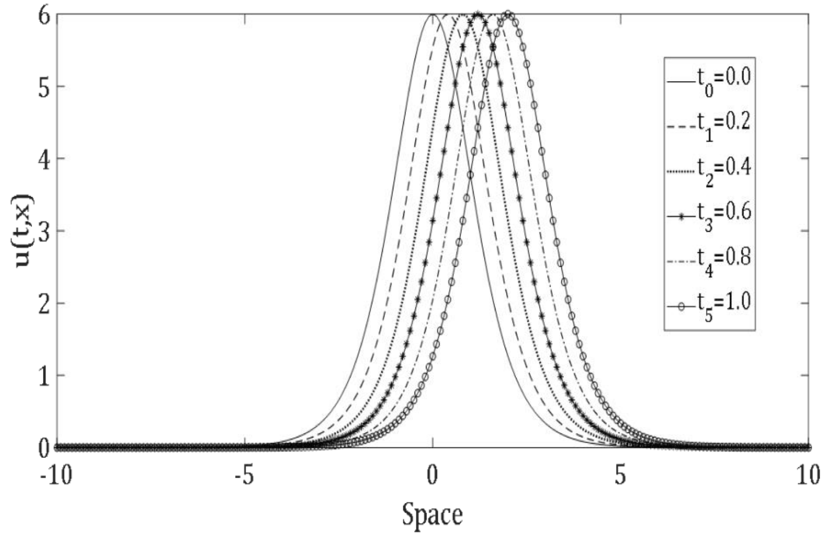

Figure 2. Analytical Solution of the KdV Equation for $\mu = 1$ and $\nu = 1$.

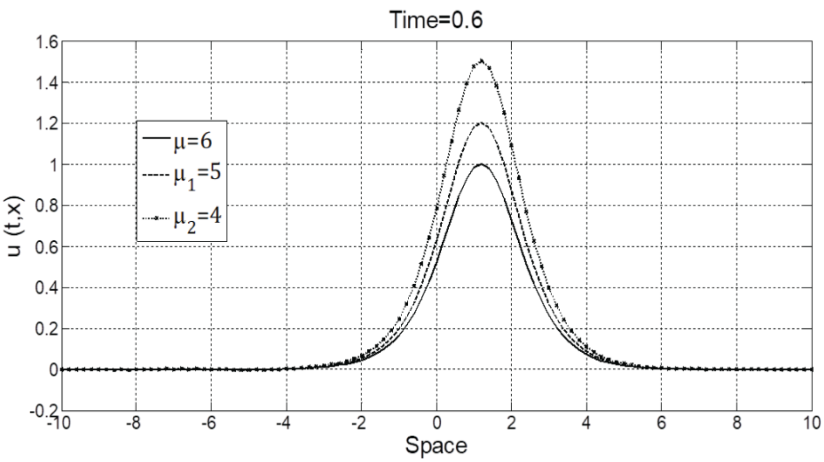

Figure 3. Effect of non-linear term at time, $t = 0.6$.

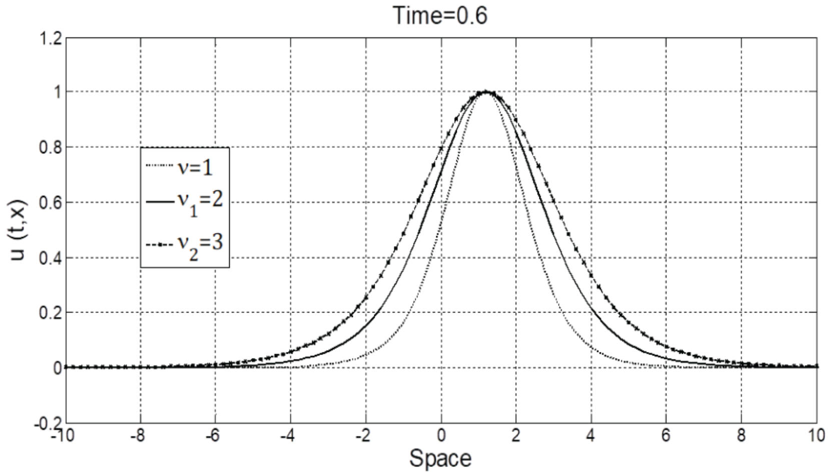

Figure 4. Effect of dispersion term at time, $t = 0.6$.

Figure 1 and Figure 2 represent the analytical solution of the KdV equation for $\mu = 6$, $\nu = 1$ and $\mu = 1$, $\nu = 1$ respectively. Figure 3 represents the effect of nonlinear coefficient where for increasing the value of nonlinear coefficient the height of the wave decreasing. Here for fixed $\nu = 1$, we consider $\mu = 4, 5, 6$ respectively. From Figure 4 it is observed that for fixed $\mu = 6$ and considering $\nu = 1, 2, 3$ (increasing) respectively, the width of the wave is spreading at the time, $t = 0.6$ which shows the effect of the dispersion term.

5. Numerical Presentation

An error estimation of the explicit finite difference schemes is presented in this section. For error estimation L_1 norm is used, defined by

$$\|e\|_1 = \frac{\|u_e - u_n\|_1}{\|u_e\|_1}$$

for all time where u_e and u_n represent exact numerical solution respectively. Initial and boundary conditions are taken from the exact solution. As for zero boundary condition [13] of the exact solution at infinity, boundary value is approximately zero on the considered domain.

Here errors are estimated for first-order non-conservative (FONC) and second-order non-conservative (SONC) forms. And then error comparing SONC form and ZK scheme for different values of Δt and Δx. In exact solution two sets of data are considered: one is $\mu = 6, \nu = 1, c = 2, \xi_0 = 0$ and the other case is $\mu = 1, \nu = 1, c = 2, \xi_0 = 0$. For both two sets of date, the numerical solution of the KdV equation for $\mu = 6, \nu = 1$ and $\mu = 1, \nu = 1$ are presented. These two cases are discussed unitedly. For numerical solution, taking $\Delta t = 0.0002$ and $\Delta x = 0.2000$ (for first order schemes) and $\Delta t = 0.0002$ and $\Delta x = 0.1000$ (for second order and Zk schemes). And for error estimation, different sets of Δt and Δx are considered, and $\mu = 6$ and $\nu = 1$are considered. Graphical representations for various cases are given in (Figures 5 - 14).

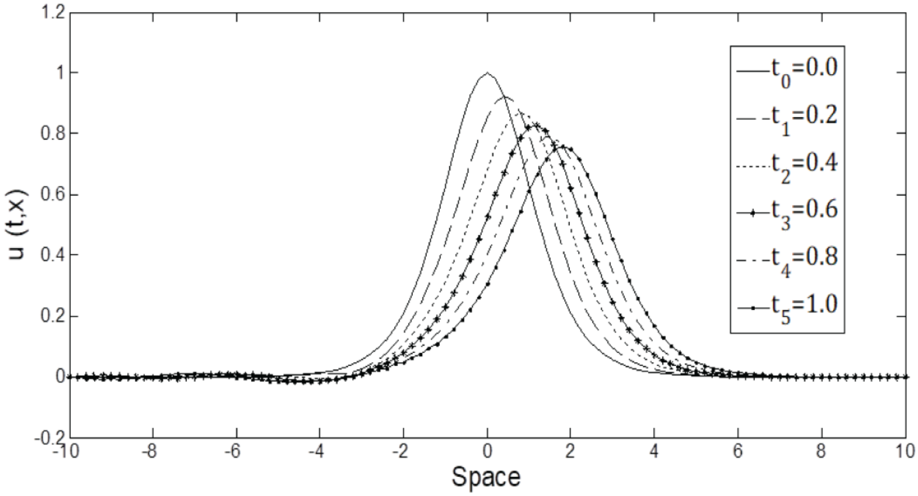

Figure 5. First order non-conservative form for $\mu = 6, \nu = 1$.

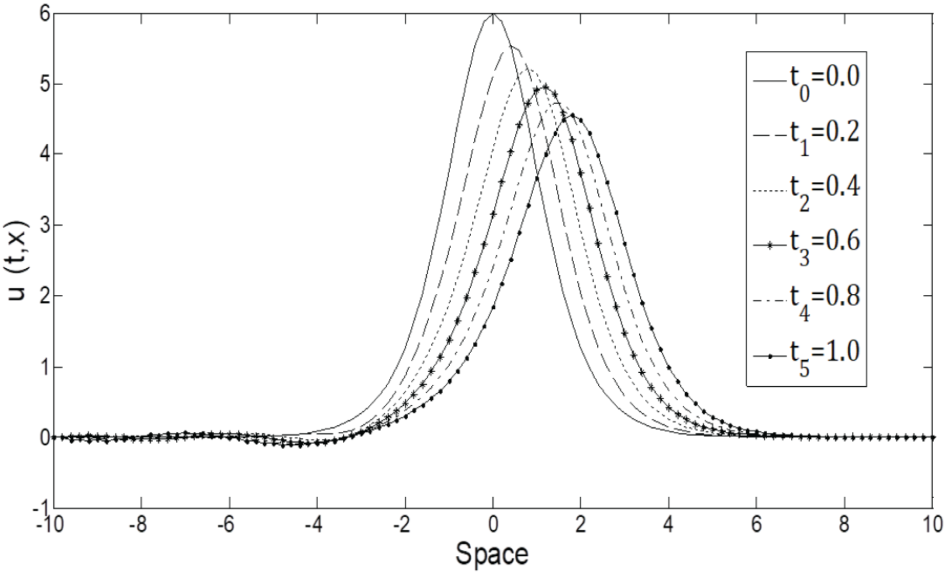

Figure 6. First order non-conservative form for $\mu = 1, \nu = 1$.

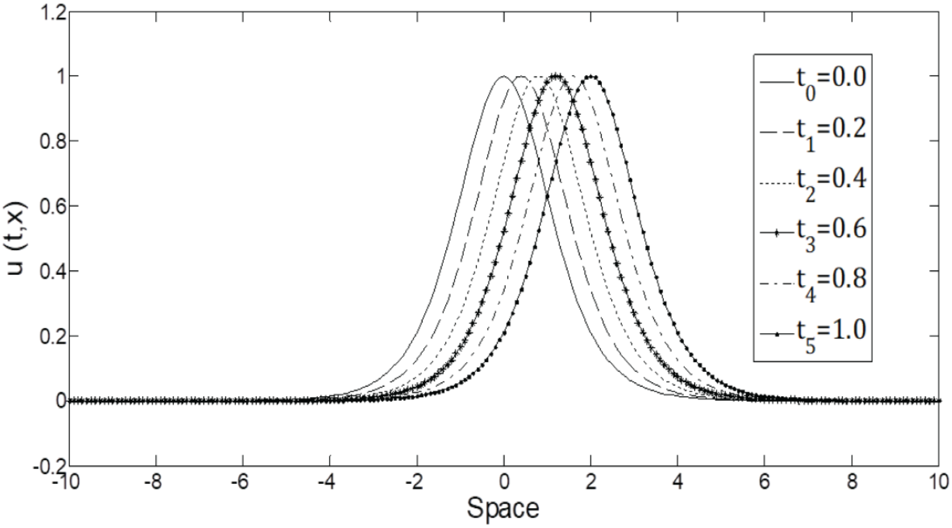

Figure 7. Second order non-conservative form for $\mu = 6, \nu = 1$.

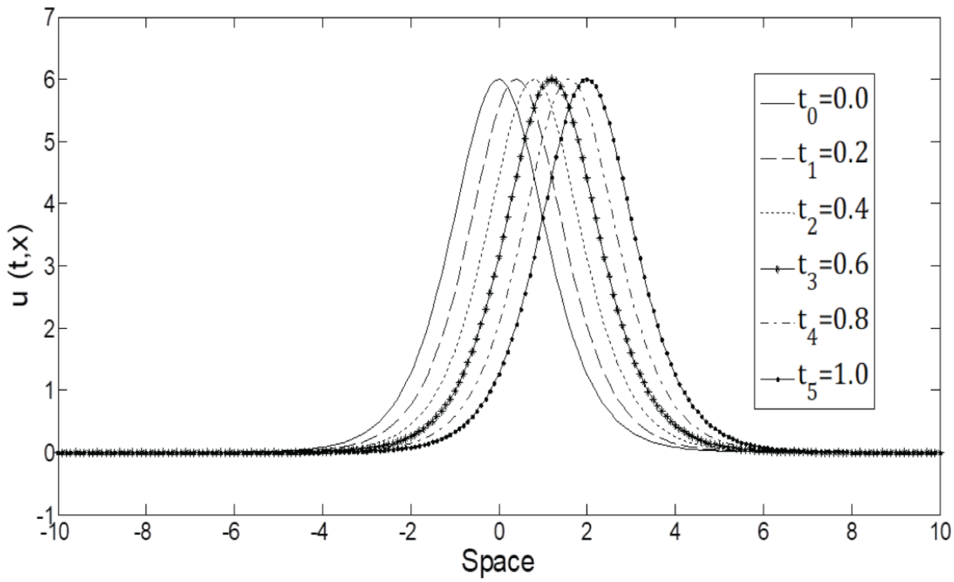

Figure 8. Second order non-conservative form for $\mu = 1, \nu = 1$.

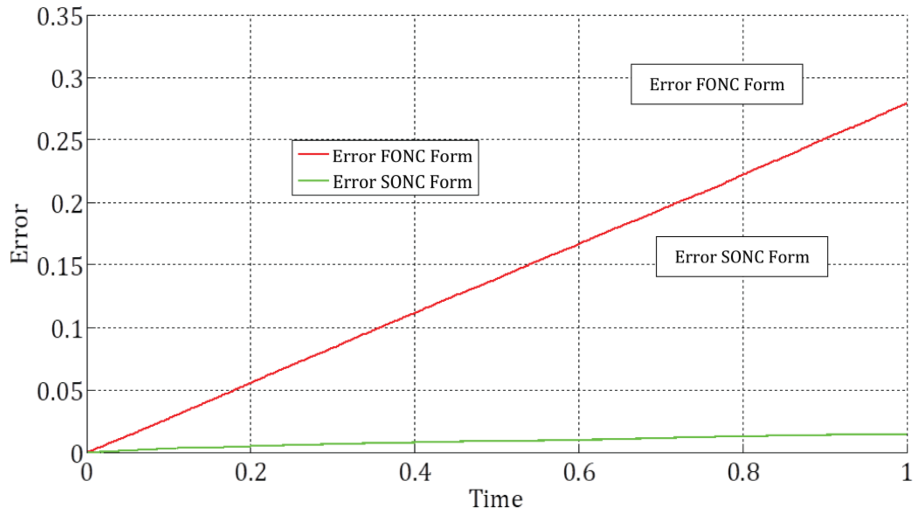

Figure 9. Error comparing of the first order non-conservative form (FONC Form) and second order non-conservative form (SONC Form) for $\mu = 6, \nu = 1$.

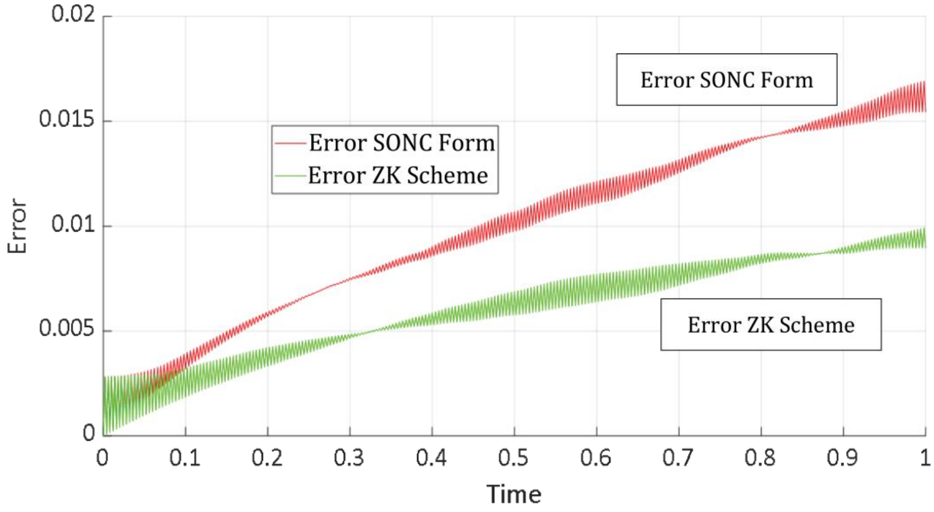

Figure 10. Error comparing of SONC Form and ZK Scheme for $\mu = 6, \nu = 1$ where $\Delta t = 0.0020$ and $\Delta x = 0.2000$.

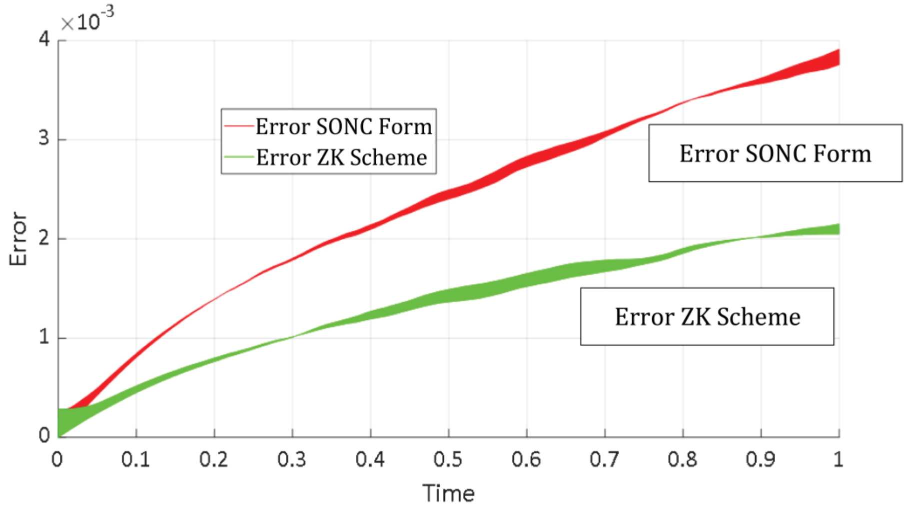

Figure 11. Error comparing of SONC Form and ZK Scheme for $\mu = 6, \nu = 1$ where $\Delta t = 0.0002$ and $\Delta x = 0.1000$.

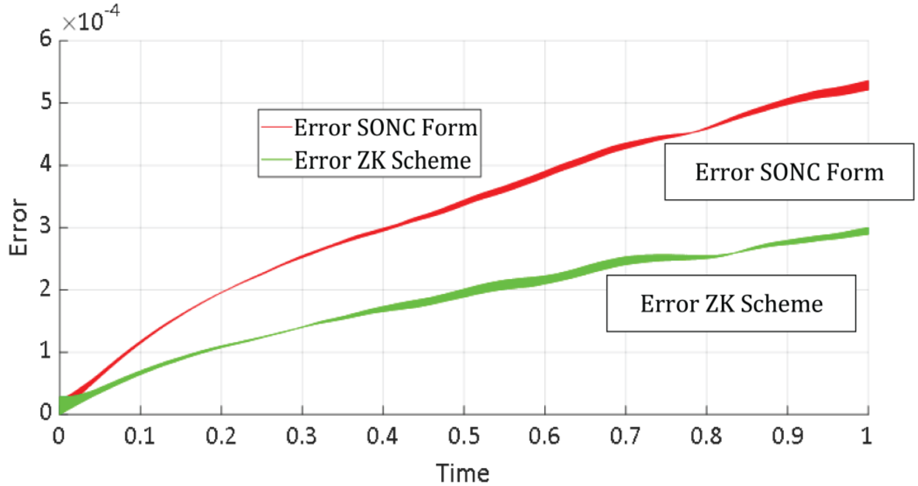

Figure 12. Error comparing of SONC Form and ZK Scheme for $\mu = 6, \nu = 1$ where $\Delta t = 0.00002$ and $\Delta x = 0.0374$.

Figure 13. Error comparing of SONC Form and ZK Scheme for $\mu = 6, \nu = 1$ where $\Delta t = 0.00001$ and $\Delta x = 0.0296$.

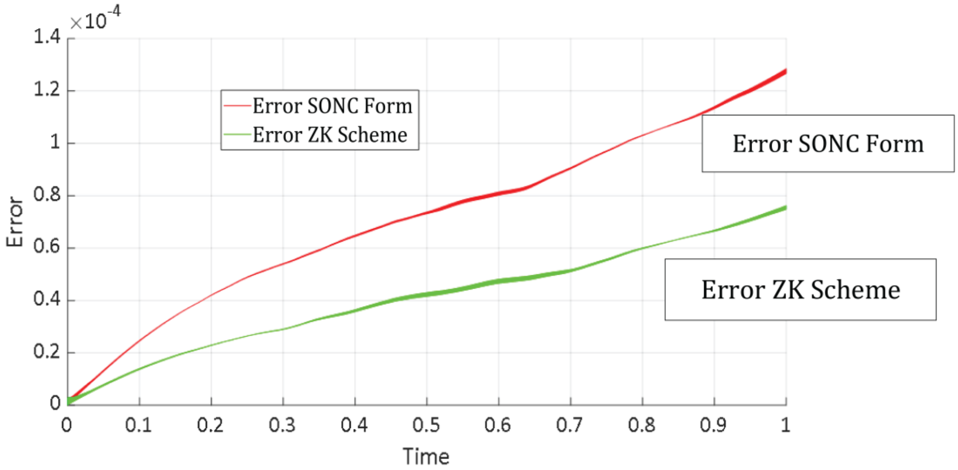

Figure 14. Error comparing of SONC Form and ZK Scheme for $\mu = 6, \nu = 1$ where $\Delta t = 0.000002$ and $\Delta x = 0.0174$.

Table 1: Relative Errors and Percentages

Figure	Δt	Δx	Relative Error of SONF at t=1	Relative Error of ZK Scheme at t=1	Difference Between Relative Error	Percentage of the Relative Error	Time Elapsed (sec)
Figure 10	0.0020	0.2000	0.0155	0.0090	0.0065	0.6488	0.995384
Figure 11	0.0002	0.1000	0.0038	0.0021	0.0017	0.1707	3.262973
Figure 12	0.00002	0.0374	5.2102e-04	2.8938e-04	2.3165e-04	0.0232	42.805090
Figure 13	0.00001	0.0296	3.2556e-04	1.8579e-04	1.3977e-04	0.0140	57.891461
Figure 14	0.000002	0.0174	1.2670e-04	7.4847e-05	5.1849e-05	0.0052	2158.678017

6. Explanation of the Graphical Representations

Figures 5 and 6 represent the numerical solution of the first order non-conservative forms for $\mu = 6, \nu = 1$ and $\mu = 1, \nu = 1$ respectively. The figures show the amplitude of the waves decreasing for increasing time and width of the waves also spreading. Both are the effects of convection and dispersion terms respectively. Figures 7 and 8 represent the numerical solution of the second order non-conservative forms for $\mu = 6, \nu = 1$ and $\mu = 1, \nu = 1$ respectively. Having glanced at two figures, waves propagate approximately same height and width, because the scheme is second order accuracy in both time and space and a solitary wave obtained by second order scheme. Figure 9 represents the Error comparing of FONC form and SONC Form for $\mu = 6, \nu = 1$. Here the error of first order scheme is so high. Because the low order of the first order scheme which is killing the accuracy of the higher order u_{xxx} approximation and the error is $o(\Delta\, x)$. In addition, first order scheme is unable to produce solitary waves. From Figure 9 we conclude that second order non-conservative is much better than first order non-conservative form.

Figure 10 to Figure 14 represent error comparison between SONC form and ZK scheme for different values of Δt and Δx . Here all those comparison are computed for $\mu = 6, \nu = 1$. In In Figure 10, Figure 11, Figure 12, Figure 13, Figure 14 we use respectively $\Delta t = 0.002,\ 0.0002,$ $0.00002,\ 0.00001,\ 0.000002$ and corresponding $\Delta x =$ 0.2, 0.1, $0.0374, 0.0296,\ 0.0174$ respectively. We observed that error of SONC form is greater than ZK scheme. Table 1 represents the summarization of the relative errors of the both schemes and mention the percentage of the error. From above qualitative observation, it is seen that the error of the as for decreasing Δt and Δx the SONC form close to ZK scheme but it take more time for decreasing Δt and Δx.

7. Conclusion

In this paper, we have studied an analytic solution of the general form of nonlinear third order KdV equation by using traveling wave solution method. Explicit finite difference schemes for the numerical solution of the KdV equation have been investigated. The stability condition for the first-order scheme using the convex combination method has been determined. Von Neumann stability analysis is performed to determine the stability condition for a second order scheme. The effects of convection and dispersion terms are being verified. The paper presents error estimations of the finite difference schemes and compared the schemes with ZK Scheme [10]. We have observed that second order conservative with non-form is close to ZK scheme.

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