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Bayesian Estimation of Fecundability for Heterogeneous Cohort of Women using Asymmetric Loss Functions

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Abstract

In this paper, we present a Bayesian approach to find the estimators of fecundability parameter for heterogeneous cohort of women population using non-linear exponential (NLINEX) and linear-exponential (LINEX) loss functions, both are asymmetric. The asymmetry brings about the shift of location of the loss function. A study of the derived estimators are also done.

Keywords and Phrases: Bayes' estimator, Loss function, Prior and posterior distributions, Fecundability parameter and Lindley's approximation.

Introduction

The two factors such as fertility rate and growth rate are important of the size and decomposition of a population. These are governed and related by the terms fecundity and fecundability. Fecundity means the potential or physiological capacity of women to produce a live birth where as fecundability is the probability of conception during a given menstrual cycle of those women who did not use any family planning method before their first conception and are sexually active.

Fecundability has an opposite relationship to the conception interval, conception delay. Conception interval and fecundability are two important and inter related fertility parameters, regarded as the most direct measures of fertility of a population. Thus the concept of fecundability is one of the principal determining factor of fertility and to human reproductive behavior in different societies.

In homogeneous women population where fecundability is assumed to be constant, the most commonly and widely applicable technique is to estimate the fecundability directly from geometric distribution. But in real life, it has seen that due to effect of many socio-economic and demographic variables, fecundability varies from women to women and hence it may be

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thought of as a random variable. In this situation, beta distribution of first kind is considered as an useful model for such a heterogeneous cohort of women population.

Suppose X be the random month of waiting to conception, then it has the conditional geometric distribution with probability mass function (p.m.f) is given by

$$P(X = x \mid \theta) = \theta(1-\theta)^{x-1}; \quad x = 1, 2, 3, \cdots, \propto, \quad 0 \le \theta \le 1, \tag{1}$$

where fecundability parameter θ assumed to be constant in homogeneous women population.

Now, if θ varies among couples, then it has following probability density function (p.d.f) as;

$$f(\theta; p, q) = \frac{1}{B(p,q)}\theta^{p-1}(1-\theta)^{q-1}; \; 0 \le \theta \le 1, \;\; p, q > 0, \tag{2}$$

where p and q are shape parameters.

As such the fecundability of a particular women which is assumed to be constant earlier from month to month may be thought of as a realization of the random variable θ. However, the estimation of fecundability in homogeneous model was studied by different authors such as Bongaarts [5], Bendel and Hua [6], Islam and Yadava [8], Goldman et. al. [10], Balakrishnan [12] and James [13] etc. Some Bayesian analysis of fecundability was done by Chowdhury and Umbach [9]. Estimation of fecundability parameter in Bayesian approach under different loss functions was also studied by Podder [3].

Now, the purpose of the paper is to find the estimators of fecundability parameter θ for heterogeneous cohort of women population in a Bayesian approach using non-linear exponential (NLINEX) and linear-exponential (LINEX) asymmetric loss functions considering the conditional geometric distribution in (1) as a fecundability model distribution and beta distribution of first kind in (2) as a prior density for θ.

The Bayes' estimators considered so far are more or less based on symmetric loss functions. But it has seen that in some decision problems, the use of symmetric loss functions might be inappropriate because of the same magnitude of error, a given positive error might be more serious than a given negative error or vice-versa. For instance, in dam construction, an underestimate of the peak water level in usually much more serious than an overestimate. Therefore, for those cases using an asymmetric loss function is more desirable.

This paper also provides a study among the Bayes' estimators of θ under non-linear exponential (NLINEX), linear-exponential (LINEX) and squared-error (SE) loss functions as well as maximum likelihood estimator (MLE) in **Section 4**.

2. Preliminaries

Let X be a random variable whose distribution depends on parameter θ and let Ω denotes the parameter space of possible values of θ. Now consider the general problem of estimating the unknown parameter θ, from the results of a random sample of n observations. Denoting the sample observations $x_1, x_{2,} \cdots, x_n$ by x, let $\hat{\theta}$ be an estimator of θ and also let $L(\hat{\theta}, \theta)$ be the loss incurred by taking the value of the parameter θ to be $\hat{\theta}$.

If $l(\theta \mid x)$ is the likelihood function of θ given the sample x and $g(\theta)$ be the prior density of θ, then combining $l(\theta \mid x)$ and $g(\theta)$, it produces the posterior distribution $P(\theta \mid x)$ though the Bayes' theorem as

$$P(\theta \mid x) = \frac{l(\theta \mid x) g(\theta)}{p(x)}, \tag{3}$$

where $p(x) = \int_\Omega l(\theta \mid x) g(\theta) d\theta$.

Hence, the Bayes' estimator $\hat{\theta}$ of θ will be a solution of the equation

$$\int_\Omega \frac{\delta L}{\delta \theta} P(\theta \mid x) = 0, \tag{4}$$

where L stands for loss function and assume that the sufficient regularities conditions prevail to permit differentiation under the sign of integral.

Here, we consider the following loss functions as;

$$\text{(i) } L_1\left(\hat{\theta},\theta\right)=\kappa\left[e^{c\left(\hat{\theta}-\theta\right)}+c\left(\hat{\theta}-\theta\right)^2-c\left(\hat{\theta}-\theta\right)-1\right];\ \kappa>0,\ c\neq 0 \tag{5}$$

$$\text{(ii) } L_2\left(\hat{\theta},\theta\right)=\kappa\left[e^{c\left(\hat{\theta}-\theta\right)}-c\left(\hat{\theta}-\theta\right)-1\right];\ \kappa>0,\ c\neq 0 \tag{6}$$

$$\text{(iii) } L_3\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2; \tag{7}$$

where $\kappa>0$ and $c\neq 0$ are scale and shape characteristics respectively.

The above loss function $L_1\left(\hat{\theta},\theta\right)$ is called non-linear exponential (NLINEX) proposed by Islam et. al. [2] and the linear-exponential (LINEX) loss function $L_2\left(\hat{\theta},\theta\right)$ introduced by Varian [7] and studied by several authors, such as Zellner [1], Christoffersen and Diebold [11] etc. For $c>0$, it penalizes an error almost exponentially, for a positive error and almost linearly for a negative error or vice-versa for $c<0$. The squared-error (SE) loss function $L_3\left(\hat{\theta},\theta\right)$ is a symmetric one.

Using (5) in (4), we have

$$c\kappa\int_\Omega\left[e^{-c\theta}e^{c\hat{\theta}}+2\left(\hat{\theta}-\theta\right)-1\right]P(\theta\mid x)d\theta=0$$

$$\Rightarrow e^{c\hat{\theta}}E_\theta\left(e^{-c\theta}\mid x\right)=1-t,$$

where $t=2E_\theta\left(\hat{\theta}-\theta\right)$.

Taking logarithm on both sides, it becomes

$$\Rightarrow c\hat{\theta}+\ln E_\theta\left(e^{-c\theta}\mid x\right)=\ln\left(1-t\right).$$

Now, assume that since Bayes' estimator is consistent, hence

$$\lim_{n\to\infty}E\left(\hat{\theta}-\theta\right)^r\to 0,\quad r\geq 2\,.$$

Expanding $\ln(1-t)$ on the basis of the assumption and neglecting 2nd and higher power of t, where $t = 2E_\theta\left(\hat{\theta}-\theta\right)$, we have

$$\hat{\theta} = -\frac{1}{(c+2)}\left[\ln E_\theta\left(e^{-c\theta} \mid x\right) - 2E_\theta(\theta \mid x)\right].$$

Therefore, the Bayes' estimator of fecundability parameter θ under the NLINEX loss function in (5) is

$$\hat{\theta}_{NL} = -\frac{1}{(c+2)}\left[\ln E_\theta\left(e^{-c\theta} \mid x\right) - 2E_\theta(\theta \mid x)\right]. \tag{8}$$

Similarly, the Bayes' estimator of θ under the LINEX and SE loss functions in (6) and (7) are as follows;

$$\hat{\theta}_{BL} = -\frac{1}{c}\ln E_\theta\left(e^{-c\theta} \mid x\right) \tag{9}$$

and $\hat{\theta}_{SE} = E_\theta(\theta \mid x)$, (10)

the mean of the posterior distribution and E_θ stands for the posterior expectation.

3. Bayes' Estimation

In this section, the Bayes' estimators of fecundability parameter θ under NLINEX and LINEX loss functions are discussed in details.

Let us consider a random sample $x = (x_1, x_2, \cdots, x_n)$ of size n drawn from (1). The likelihood function of θ for the given sample observation x is

$$l(\theta \mid x) = \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i - n}$$

$$= \theta^n (1-\theta)^{y-n}, \tag{11}$$

where $y = \sum_{i=1}^{n} x_i$.

The maximum likelihood estimator (MLE) of θ is $\frac{1}{\bar{x}}$, as sample mean $\bar{x}$.

It is noted that the part of the likelihood function which is relevant to a Bayesian inference on the unknown parameter θ is $\theta^{n}(1-\theta)^{y-n}$.

For the problem under consideration let us assume that the conjugate prior density for fecundability parameter θ as

$$g(\theta)=\frac{1}{B(p,q)}\theta^{p-1}(1-\theta)^{q-1};\ 0\le\theta\le 1,\ p>0,\ q>0, \tag{12}$$

where p and q may be obtained from the sample data in any classical approach like method of maximum likelihood and method of moments if unknown.

The relation in (12) is simply a member of the beta family of distributions. The advantage of taking the prior density to be conjugate lies in the fact that the likelihood function $l(\theta \mid x)$, the prior density $g(\theta)$ and the posterior density $P(\theta \mid x)$ are all of the same functional form thus ensuring mathematical tractability.

By combining (11) and (12), we obtain the posterior density of θ as

$$P(\theta \mid x)\propto l(\theta \mid x)g(\theta)$$

$$\Rightarrow P(\theta \mid x)\propto \theta^{n+p-1}(1-\theta)^{y-n+q-1}.$$

This implies that the posterior distribution of fecundability parameter θ for the given sample observation $x=(x_1, x_{2,} \cdots, x_n)$ is

$$P(\theta \mid x)=\frac{1}{B(n+p,\ y-n+q)}\theta^{n+p-1}(1-\theta)^{y-n+q-1};\ 0\le\theta\le 1,\ p>0,\ q>0, \tag{13}$$

which follows that $\theta \sim Beta(n+p, y-n+q)$. The posterior fecundability model is an updated version of our prior knowledge about θ in light of the observed random sample.

The mean of the posterior distribution (13) is

$$E(\theta \mid x) = \frac{n+p}{y+p+q} \ . \tag{14}$$

To obtain Bayes' estimators of fecundability parameter θ under non-linear exponential (NLINEX) and linear-exponential (LINEX) loss functions using (8) and (9) now, we shall find the posterior expectation of $e^{-c\theta}$ that is the expectation of $e^{-c\theta}$ with respect to the posterior distribution in (13) as

$$E_\theta\left[e^{-c\theta} \mid x\right] = \int_\Omega e^{-c\theta} P(\theta \mid x) d\theta$$

$$= \frac{1}{B(n+p, y-n+q)} \int_0^1 e^{-c\theta} \theta^{n+p-1} (1-\theta)^{y-n+q-1} d\theta$$

$$= \frac{1}{B(n+p, y-n+q)} \int_0^1 \exp\left[-c\theta + (n+p-1)\ln\theta + (y-n+q-1)\ln(1-\theta)\right] d\theta \ . \tag{15}$$

The integral in (15) is not readily solvable theoretically. For evaluating this integral we shall use the technique of Lindley's [4] approximation.

Approximate Bayes' estimator of θ using Lindley's approximation

According to Lindley and ratio of integrals of the form

$$I(X) = \frac{\int u(\theta) e^{L(\theta)+\rho(\theta)} d\theta}{\int e^{L(\theta)+\rho(\theta)} d\theta} \ , \tag{16}$$

can approximately be evaluated as

$$I(X) = u(\hat{\theta}) + \frac{1}{2}\left[\left\{u''(\hat{\theta}) + 2u'(\hat{\theta})\rho'(\hat{\theta})\right\} + L'''(\hat{\theta})u'(\hat{\theta})\hat{\sigma}^2(\hat{\theta})\right]\hat{\sigma}^2(\hat{\theta}), \tag{17}$$

where

$u(\theta) =$ a function of θ

$L(\theta) =$ log-likelihood function of θ

$\rho(\theta) =$ logarithm of a prior distribution of θ

$\hat{\theta} =$ maximum likelihood estimator (MLE) of θ

$$u'(\hat{\theta}) = \frac{\delta}{\delta\theta} u(\theta)\Big|_{\theta=\hat{\theta}} \quad , \quad u''(\hat{\theta}) = \frac{\delta^2}{\delta\theta^2} u(\theta)\Big|_{\theta=\hat{\theta}} \quad , \quad \rho'(\hat{\theta}) = \frac{\delta}{\delta\theta} \rho(\theta)\Big|_{\theta=\hat{\theta}} \quad ,$$

$$L''(\hat{\theta}) = \frac{\delta^2}{\delta\theta^2} L(\theta)\Big|_{\theta=\hat{\theta}} \; , \; L'''(\hat{\theta}) = \frac{\delta^3}{\delta\theta^3} L(\theta)\Big|_{\theta=\hat{\theta}} \text{ and } \sigma^2(\hat{\theta}) = -\frac{1}{L''(\hat{\theta})} .$$

Evaluation of Bayes' estimator of θ by Lindley's approximation

The posterior expectation of $e^{-c\theta}$ that is $E_\theta\left[e^{-c\theta} \mid x\right]$ rewritten as

$$\hat{E}_\theta\left[e^{-c\theta} \mid x\right] = \int_\Omega e^{-c\theta} P(\theta \mid x) d\theta$$

$$= \frac{\int_\Omega u(\theta) e^{L(\theta)+\rho(\theta)} d\theta}{\int_\Omega e^{L(\theta)+\rho(\theta)} d\theta} ,$$

with

$$u(\theta) = e^{-c\theta}$$

$$L(\theta) = n \ln \theta + (y-n)\ln(1-\theta)$$

$$\rho(\theta) = -\ln[B(p,q)] + (p-1)\ln \theta + (q-1)\ln(1-\theta)$$

$$u'(\theta) = -ce^{-c\theta}$$

$$u''(\theta) = c^2 e^{-c\theta}$$

$$\rho'(\theta) = \frac{p-1}{\theta} - \frac{q-1}{1-\theta}$$

$$L''(\theta) = -\frac{n}{\theta^2} - \frac{y-n}{(1-\theta)^2}$$

$$L'''(\theta) = \frac{2n}{\theta^3} - \frac{2(y-n)}{(1-\theta)^3}$$

and $\sigma^2(\theta) = \dfrac{1}{\dfrac{n}{\theta^2} + \dfrac{y-n}{(1-\theta)^2}}$.

Using the above expression in (17), the Lindley's approximation gives

$$I(X) = E\left(e^{-c\theta} \mid x\right) = e^{-c\theta} + \frac{1}{2} c e^{-c\theta} \left[\frac{\left\{ c - 2\left(\frac{p-1}{\theta} - \frac{q-1}{1-\theta} \right) \right\}}{\left\{ \frac{n}{\theta^2} + \frac{y-n}{(1-\theta)^2} \right\}} - \frac{2\left\{ \frac{n}{\theta^3} - \frac{y-n}{(1-\theta)^3} \right\}}{\left\{ \frac{n}{\theta^2} + \frac{y-n}{(1-\theta)^2} \right\}^2} \right].$$

Therefore,

$$\hat{E}\left(e^{-c\theta} \mid x\right) = e^{-c\hat{\theta}} \left[1 + \frac{c}{2} . \frac{\left\{ c - 2\left(\frac{p-1}{\hat{\theta}} - \frac{q-1}{1-\hat{\theta}} \right) \right\}}{\left\{ \frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2} \right\}} - \frac{2\left\{ \frac{n}{\hat{\theta}^3} - \frac{y-n}{(1-\hat{\theta})^3} \right\}}{\left\{ \frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2} \right\}^2} \right],$$

and hence the logarithmic posterior expectation of $e^{-c\theta}$ that is $\ln E_\theta\left[e^{-c\theta} \mid x\right]$ as follows,

$$\ln \hat{E}\left(e^{-c\theta} \mid x\right) = -c\hat{\theta} + \ln \left[1 + \frac{c}{2} . \frac{\left\{ c - 2\left(\frac{p-1}{\hat{\theta}} - \frac{q-1}{1-\hat{\theta}} \right) \right\}}{\left\{ \frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2} \right\}} - \frac{2\left\{ \frac{n}{\hat{\theta}^3} - \frac{y-n}{(1-\hat{\theta})^3} \right\}}{\left\{ \frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2} \right\}^2} \right]. \quad (18)$$

Now using (18) in (8), the Bayes' estimator of the fecundability parameter θ under NLINEX loss function is

$$\hat{\theta}_{NL} = \frac{c\hat{\theta}}{(c+2)} - \frac{1}{(c+2)}\ln\left[1 + \frac{c}{2}.\frac{\left\{c - 2\left(\frac{p-1}{\hat{\theta}} - \frac{q-1}{1-\hat{\theta}}\right)\right\}}{\left\{\frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2}\right\}} - \frac{2\left\{\frac{n}{\hat{\theta}^3} - \frac{y-n}{(1-\hat{\theta})^3}\right\}}{\left\{\frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2}\right\}^2}\right] + \frac{2(n+p)}{(c+2)(y+p+q)} \qquad (19)$$

Similarly, using (18) and (9), the Bayes' estimator of θ under LINEX loss function is

$$\hat{\theta}_{BL} = \hat{\theta} - \frac{1}{c}\ln\left[1 + \frac{c}{2}.\frac{\left\{c - 2\left(\frac{p-1}{\hat{\theta}} - \frac{q-1}{1-\hat{\theta}}\right)\right\}}{\left\{\frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2}\right\}} - \frac{2\left\{\frac{n}{\hat{\theta}^3} - \frac{y-n}{(1-\hat{\theta})^3}\right\}}{\left\{\frac{n}{\hat{\theta}^2} + \frac{y-n}{(1-\hat{\theta})^2}\right\}^2}\right], \qquad (20)$$

where $n > 0$, $p > 0$, $q > 0$, $c \neq 0$ and $y = \sum_{i=1}^{n} x_i$.

Noting that in (19) and (20), $\hat{\theta}$ is a maximum likelihood estimator (MLE) of fecundability parameter θ which is the reciprocal of a sample mean $\bar{x}$ defined earlier.

The Bayes' estimator under squared-error (SE) loss function in (10) is

$$\hat{\theta}_S = \frac{n+p}{y+p+q} \; , \qquad (21)$$

the mean of the posterior distribution in (13).

Using (20) and (21), the Bayes' estimator under NLINEX loss function may be rewritten as

$$(c+2)\hat{\theta}_{NL} = c\hat{\theta}_{BL} + 2\hat{\theta}_S \; ,` \qquad (22)$$

which is a linear combination of $\hat{\theta}_{BL}$ and $\hat{\theta}_S$, Bayes' estimators under linear-exponential (LINEX) and squared-error (SE) loss functions respectively.

4. Conclusion

In this Section, an attempt has been made to study among the Bayes' estimators of fecundability parameter $\hat{\theta}_{NL}$, $\hat{\theta}_{BL}$ and $\hat{\theta}_S$ under non-linear exponential (NLINEX), linear-exponential (LINEX) and squared-error (SE) loss functions respectively as well as $\hat{\theta}$, a classical maximum likelihood estimator (MLE). Therefore, we may conclude that as follows;

(i) The maximum likelihood estimator (MLE) is a reciprocal of sample mean $\bar{x}$ that is $\hat{\theta} = \frac{1}{\bar{x}}$.

(ii) Bayes' estimators under non-linear exponential (NLINEX), linear-exponential (LINEX) and squared-error (SE) loss functions are related by the equation $(c+2)\hat{\theta}_{NL} = c\hat{\theta}_{BL} + 2\hat{\theta}_S$, and hence $\hat{\theta}_{NL} = c\hat{\theta}_{BL}/(c+2) + 2\hat{\theta}_S/(c+2)$, where $\hat{\theta}_{NL}$, $\hat{\theta}_{BL}$ and $\hat{\theta}_S$ are Bayes' estimators under NLINEX, LINEX and SE loss functions respectively.

(iii) Among the three Bayes' estimators, if any two of them are known then rest can be evaluated by (ii).

(iv) The Bayes' estimator under non-linear exponential (NLINEX) loss function is a linear combination of Bayes' estimators under linear-exponential (LINEX) and squared-error (SE) loss functions.

(v) The Bayes' estimator under non-linear exponential (NLINEX) loss function which is an asymmetric one, a linear combination of Bayes' estimators of asymmetric linear-exponential (LINEX) and symmetric squared-error (SE) loss functions respectively.

(vi) As c tends to infinity both Bayes' estimators under NLINEX and LINEX loss functions identical.

(vii) As $c \to \propto$ Bayes' estimator under LINEX loss function $\hat{\theta}_{BL}$ tends to $\hat{\theta}$, a maximum likelihood estimator (MLE).

(viii) As $p \to 0$ and $q \to 0$, then Bayes' estimator under squared-error (SE) loss function $\hat{\theta}_S$ tends to MLE.

(ix) As $n \to \infty$, $\theta_S \to \frac{1}{\bar{x}}$, which implies that Bayes' estimator under SE loss function tends to classical maximum likelihood estimator (MLE).

(x) Bayes' estimator under NLINEX loss function gives a general form for odd as well as even number c, as follows;

$$\hat{\theta}_{NL} = \begin{cases} \left(i\hat{\theta}_{BL} + 2\hat{\theta}_S\right) / (i+2), & i = 1, 3, 5. \cdots, \textit{ when } c \textit{ is an odd}; \\ \left(j\hat{\theta}_{BL} + 2\hat{\theta}_S\right) / (j+2), & j = 2, 4, 6. \cdots, \textit{ when } c \textit{ is an even}. \end{cases}$$

(xi) One of the important features observed for the odd and even numbers of c is that the Bayes' estimator $\hat{\theta}_{NL}$ provides a weighted average of the Bayes' estimators $\hat{\theta}_{BL}$ and $\hat{\theta}_S$ respectively.

References

[1] A. Zellner, Bayesian Estimation and Prediction using Asymmetric Loss Function, Journal of American Statistical Association, 1986, 81, 446.

[2] A. F. M Saiful Islam, M. K. Roy and M. Masoom Ali, A Non-Linear Exponential Loss Function in Bayesian Analysis, Journal of Korean Data & Information Science Society, 2004, 15(4), 899.

[3] C. K. Podder, Estimation of Fecundability in Bayesian Approach using Different Loss Functions, Pakistan Journal of Statistics, 2019, 35(1), 1.

[4] D. V. Lindley, "Approximate Bayesian Methods", Bayesian Statistics, Balencia, Spain, University Press, 1980.

[5] J. Bongaarts, A Method For The Estimation of Fecundability, Demography, 1975, 12, 645.

[6] J. P. Bendel and C. I. Hua, An Estimate of The Natural Fecundability of Ratio Curve, Social Biology, 1978, 25(3), 210.

[7] H. R. Varian, A Bayesian Approach to Real State Assessment in Studies in Bayesian Econometrics and Statistics in Honour of L. J. Savage, eds, S. E. Fienberg and Zellner, Amsterdam, North Holland Publishing Co., 1975, 195.

[8] M. M. Islam and R. C. Yadava, Esimation of Fecundability: Levels and Trends in Bangladesh, Journal of Family Welfare, 1997, 43(3), 13.

[9] M. R. Chowdhury and D. Umbach, Some Bayesian Analysis of Fecundability, Pakistan Journal of Statistics, 2012, 28(3), 293.

[10] N. Goldman, C. F. Westoff and L. E. Paul, Estimation of Fecundability From Survey Data, Studies in Family Planning, 1985, 16, 232.

[11] P. Christoffersen and F. X. Diebold, Optimal Prediction under Asymmetric Loss, Econometric Theory, 1997, 13, 808.

[12] T. R. Balakrishnan, Probability of Conception, Conception Delay and Estimates of Fecundability in Rural and Semi-urban Areas of Certain Latin American Countries, Social Biology, 1979, 26(3), 226.

[13] W. H. James, Estimates of Fecundability, Population Studies, 1963, 17(1), 57.